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ON NUMERICAL INTEGRATION OF RAYS AND WAVEFRONT PROPAGATION.(U)

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ON NUMERICAL INTEGRATION OF RAYS  
AND WAVEFRONT PROPAGATION

⑨  
Project M-153, Technical Note Five no. 5

⑪  
November 1968

⑫ 32p.

⑩  
Prepared by: M. M. Holl

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Under Contract No. N00228-68-C-2406

For the Commanding Officer  
Fleet Numerical Weather Central  
Naval Postgraduate School  
Monterey, California

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## I Introduction

### A. Integration of Single Rays

We are dealing with the ray approach to mapping sound propagation in the sea.

We are considering the problem in which the speed of sound,  $c$ , is variable over two dimensions, the depth,  $z$ , and a horizontal coordinate,  $x$ . With no refraction normal to the  $x, z$  plane, a ray directed in the plane remains in the plane.

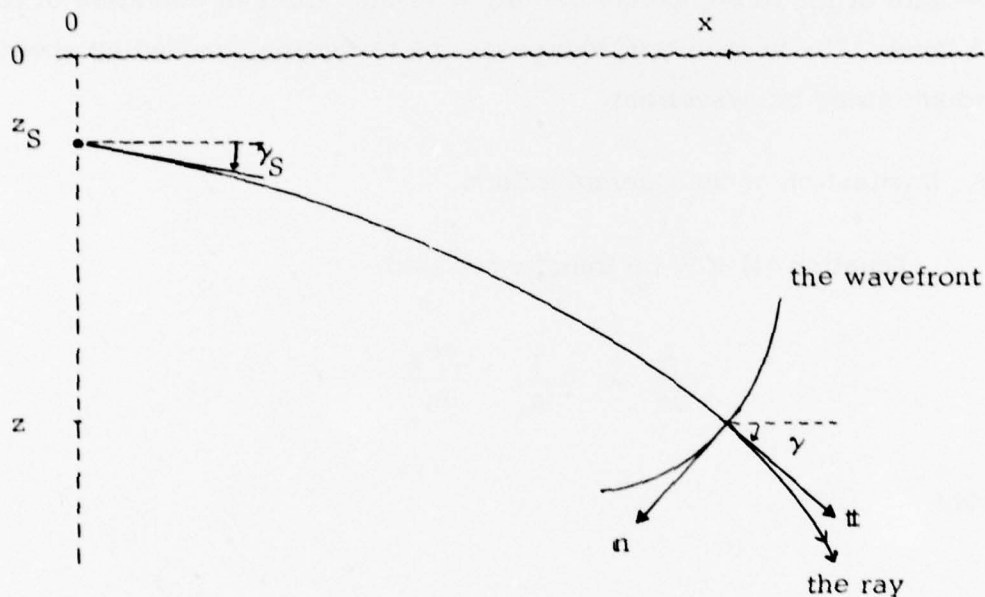


Fig. 1 A ray is the path of an element of wavefront

Each element of wavefront is instantaneously propagating along its normal, in the positive direction denoted by unit vector  $\mathbf{t}$ . A ray may be defined as the path, or locus, of an element of wavefront. The path is bent by refraction. The governing equation may be expressed by

$$\frac{d\gamma}{ds} = - \frac{\cos \gamma}{R - z} - \frac{1}{c} \frac{\partial c}{\partial n} \quad (1)$$

The notation is identified with reference to Fig. 1: The direction  $\mathbf{t}$  is represented by the angle  $\gamma$  measured clockwise positive from the horizontal. Linear distance measure along the ray is denoted by  $s$ . Linear measure normal to the ray, and hence tangent to the wavefront, along unit vector  $\mathbf{n}$  is denoted by  $n$ . The unit vector  $\mathbf{n}$  is clockwise normal to  $\mathbf{t}$  by sign convention.

The first term on the right hand side of Eq. (1) is due to coordinate curvature of the reference horizontal;  $R$  is the radius of curvature of local sea level. The second term expresses the refraction, caused by speed gradient along the wavefront.

#### B. Elimination of the Curvature Term

Equation (1) may be transformed into

$$\frac{d\gamma}{ds} = - \frac{1}{c_*} \frac{\partial c_*}{\partial n} \quad (2)$$

where

$$c_* \equiv \frac{\kappa}{R - z} c \quad (3)$$

and  $\kappa$  is an arbitrary constant. We choose to make  $\kappa = R$ .

The factor  $R/(R - z)$  ranges from 1 at the surface to 1.001 at a depth of about 6 km. Over this range of depth the factor increases the sound speed almost linearly with depth by zero at the surface to about 1.5 meters per second at 6 km. The significance for including the curvature effect

may be judged in the context of the variability of the sound-speed structure and the accuracy with which it may be specified from available information. In any case the curvature is a bias which may readily be included by modifying the speed according to Eq. (3).

The linear argument,  $s$ , along a ray may be replaced by travel time,  $t$ , according to

$$ds = c dt \quad (4)$$

The use of  $c_*$  in place of  $c$  results in a pseudo travel time

$$ds = c_* dt_* \quad (5)$$

$$dt_* = \frac{R - z}{R} dt \quad (6)$$

The modification is slight.

In all that follows we shall consistently use the modified speed, and, in the few places it appears, the modified travel time. Hence we can omit the subscript asterisk without confusion. The ray tracing is governed by

$$\frac{d\gamma}{ds} = - \frac{1}{c} \frac{\partial c}{\partial n} \quad (7)$$

### C. The Specific Wavefront Length

An individual ray may be identified at a reference point denoted by  $S$  in Fig. 1, where the orientation of the particular wavefront element is defined by  $\gamma_S$ . The wavefront element also has two dimensions associated

with its stretching (wavefront divergence) or shrinking (wavefront convergence) and reversal (wavefront folding). We denote the specific wavefront length in the  $x, z$  plane by  $L$ , and shall not concern ourselves, for present purposes, with the lateral dimension of the wavefront.

The derivation of the governing equation for  $L$  along the ray is given in Reference [1]. This governing equation may be expressed by

$$\frac{d^2 L}{ds^2} = \frac{1}{c} \frac{\partial c}{\partial s} \frac{dL}{ds} - \frac{L}{c} \frac{\partial^2 c}{\partial n^2} \quad (8)$$

The specific wavefront length may be initialized (normalized) at the point  $S$  by specification of

$$L_S$$

Initialization also requires specification of the curvature of the wavefront at  $S$ , by specification of

$$(dL/ds)_S$$

Equation (8) is for single-ray integration of wavefront spreading.

#### D. Depiction of Wavefront Propagation

The ray approach may be used to depict the propagation of wavefronts by integration of the traces of sufficient rays to adequately depict the wavefront continuum at all positions in its propagation. Such complete mapping of the wavefront propagation encompasses the wavefront spreading, shrinking and foldings, and implicitly includes full determination of  $L$ .

Independent integration, along some ray, of  $L$  by a numerical analogue of Eq. (8) thus yields another determination of  $L$ . Because of differences in truncation errors between the two independent numerical integrations--rays by numerical analogue of Eq. (7) and  $L$  by numerical analogue of Eq. (8)--the results may not be consistent. For complex sound distributions or badly designed numerical analogues the results may not even match in areas of interest. In any event, because truncation errors compound with length of the independent integrations, the match deteriorates with distance covered by the rays.

Of pragmatic interest is the propagation from a point source. Let  $S$  of Fig. 1 be the location of the point source. Then each ray is defined by its emission angle,  $\gamma_S$ , and, for all rays,

$$L_S \equiv 0 \quad (9)$$

$$(dL/ds)_S \equiv 1. \quad (10)$$

Any point in the medium may be traversed by one or more rays, giving, in association, to that point one or more values of  $\gamma_S$ . Such multivaluedness is produced by foldings of the wavefront. The wavefront remains continuous if we include the foldings in our concept of continuity, and, correspondingly, the field of  $\gamma_S$ , while multivalued, is also continuous. Thus multivalued areas are separable into superimposed families of continuous  $\gamma_S$  distributions. For each family the ascendent of  $\gamma_S$  is tangent to the wavefront in the direction of increasing source emission angle. This direction is  $+n$  or  $-n$  depending on the even-or-oddness of the number of foldings the wavefront element has undergone in its propagation from the source.

For a point source the specific wavefront length,  $L$ , may be diagnosed from full ray depiction, throughout the insonified medium, by

$$\frac{m}{L} = \nabla \gamma_S \quad (11)$$

for each family. Accordingly, and alternately but consistent with our earlier definition,  $L$  may be defined as the specific wavefront length per unit radian of emission at the point source. This definition is restricted to insonification by a point source, whereas the earlier definition is general.

#### E. Preference for Consistency

In mapping the sound propagated from a point source we have available two methods for determining the multi-valued field of  $L$ . The first method involves the design and application of a numerical analogue for the ray tracing equation, Eq. (7), and for diagnosis of the ray spacing by Eq. (11). The second method involves the design and application of a numerical analogue for Eq. (8) for integration along each ray obtained by analogue of Eq. (7). Unless the numerical analogue for Eq. (8) can be, and is, designed in a form consistent with the ray tracing, truncation errors will manifest quite differently. The disparity will generally grow with range and with the complexity of the sound-speed distribution and the ocean-bottom topography.

Such independence in truncation error may be deemed useful as indicator of the range limits imposed by truncation errors overwhelming resolution. However we dismiss this use because the likelihood is that the ray spacing would be more accurate, in general.

We deem it desirable to compute specific wavefront length,  $L$ , rather than to rely on spacing between rays of finite separation and diverse paths. And we deem it desirable that  $L$  be consistent with the distribution of rays.

We consider the information to be complementary. Continuity in  $L$  from ray to ray is an indicator of the adequacy of the resolution.

The ray to ray locus of  $L = 0$  locates caustics. Caustics generally do not lie at the intersection point of two rays of finite separation in  $\gamma_s$  but occur at the differential crossing or folding. Propagation loss by diffraction depends on the geometry of the propagation with strong dependence on the occurrence and orientation of caustics.

#### F. Objective

Our objective is to develop a numerical scheme for ray-consistent determination of the specific wavefront length distribution. This distribution generally becomes more multivalued with range from the source, not only by internal foldings in sound channels but also by reflections at sea surface and bottom.

This endeavour involves considerations which include

- maximizing accuracy in ray tracing and ray spacing for the prescribed sound speed distribution,
- simplifying the integrations for the purpose of reducing calculations for the same resolution yield,
- exploiting simplifications which occur when sound speed,  $c$ , is a function only of depth,  $z$ , and
- realizing consistency as horizontal variability in  $c$  vanishes.

We proceed by expanding on aspects of single-ray integrations and simplifying the equations for when speed is a function of depth only. We then return to designing the object scheme.

## II Single Ray Spreading Integration

### A. Transformations of the Governing Equation

Equation (8) is a form of the differential relationship which governs the specific wavefront length,  $L$ , along a ray. The motivation for transforming this equation into other forms includes revealing the character of the equation, the significances in speed-of-sound distributions, and insights for designing improved numerical analogues.

Special treatment of the equation is required for use in integrating through discontinuities in speed distribution at interfaces and at surface and bottom reflections. This development follows in Subsection B.

We begin the transformations by introducing  $K \equiv \omega n c$ .

$$\frac{d^2 L}{ds^2} = \frac{\partial K}{\partial s} \frac{dL}{ds} - L \left\{ \frac{\partial^2 K}{\partial n^2} + \left( \frac{\partial K}{\partial n} \right)^2 \right\} \quad (12)$$

Keep in mind that  $s$  is along the curved path of the ray and  $n$  is along a straight line normal to the ray. At a point along the ray we can write

$$\nabla^2 K \equiv \partial^2 K / \partial n^2 + \partial^2 K / \partial s_*^2 \quad (13)$$

where  $s_*$  is along a straight line tangent to the ray. We also require the operator transformation

$$\begin{aligned} \frac{\partial^2}{\partial s^2} &\equiv \mathbf{t} \cdot \nabla (\mathbf{t} \cdot \nabla) \equiv (\mathbf{t} \cdot \nabla)^2 + \mathbf{t} \cdot \nabla \mathbf{t} \cdot \nabla \\ &= \frac{\partial^2}{\partial s_*^2} + n \frac{d\gamma}{ds} \cdot \nabla = \frac{\partial^2}{\partial s_*^2} - \frac{\partial K}{\partial n} \frac{\partial}{\partial n} \end{aligned} \quad (14)$$

In the last step we have used Eq. (7). Accordingly we have the transformation

$$\frac{\partial^2 K}{\partial s^2} = \frac{\partial^2 K}{\partial s_\star^2} - \left( \frac{\partial K}{\partial n} \right)^2 \quad (15)$$

By using Eqs. (13) and (15) we transform Eq. (12) into

$$\frac{d^2 L}{ds^2} = \frac{\partial K}{\partial s} \frac{dL}{ds} - L \left\{ \nabla^2 K - \frac{\partial^2 K}{\partial s^2} \right\} \quad (16)$$

This form is significant because the directional operator,  $(n \cdot \nabla)^2$ , of Eq. (8) has been replaced in favor of an isotropic operator,  $\nabla^2$ , and differentials along the ray.

In Reference [2] we have transformed the governing equation into the form

$$\frac{d^2 Q}{dt^2} = -c^2 \nabla^2 K Q \quad (17)$$

where  $Q = L/c$  and travel time,  $t$ , replaces  $s$  as measure along the ray according to  $ds = c dt$ . This is the simplest form we have been able to deduce. It enabled us in Reference [2] to determine that  $Q$  is reciprocal between a point source and a receiver, for any ray path.

Initializations, at a point source, are given by Eqs. (9) and (10) and their transforms:

$$Q_S = 0 ; (dQ/ds)_S = 1/c ; (dQ/dt)_S = 1 \quad (18)$$

## B. Special Treatment at Discontinuities and Reflections

In describing the speed distribution in the sea it may be convenient or suitable to include interfaces where the speed gradient abruptly changes. For present purposes we shall treat such interfaces as horizontal. Ray tracing by Eq. (7) requires only that the speed,  $c$ , be single-valued and continuous (zero-order continuity). It can be integrated through interfaces of abrupt change in speed gradient (first-order discontinuity). However Eq. (8) cannot be integrated through such interfaces; the spreading rate,  $dL/ds$ , abruptly changes.

Reflections at the sea surface or ocean bottom, where, in general, we wish to allow  $\partial c/\partial z \neq 0$ , also involve effective passage through kinks in the speed profile. At reflections we have the additional consideration that  $L$ , in accordance with Eq. (11), flips sign.

The expressions required for carrying the integration of Eq. (8) through interfaces and reflections have been derived in Reference [3]. This was done by applying Eq. (8) to a transition layer between levels of differing specified gradient; letting the thickness of the layer shrink to zero in the analysis gave the desired expression.

$$\left[ \frac{dL}{ds} \right]_+ = \left[ \frac{dL}{ds} \right]_- - \frac{L}{c} \frac{\cos^2 \gamma}{\sin \gamma} \left\{ \left[ \frac{\partial c}{\partial z} \right]_+ - \left[ \frac{\partial c}{\partial z} \right]_- \right\} \quad (19)$$

At reflections the sign of  $L$  is flipped.

$$L_R = -L_I \quad (20)$$

The subscript  $I$  is used to denote incident value and the subscript  $R$  a reflected value. The spreading rate changes according to

$$\left[ \frac{dL}{ds} \right]_R = - \left[ \frac{dL}{ds} \right]_I + \frac{2L_R}{c} \frac{\cos^2 \gamma_I}{\sin \gamma_I} \frac{\partial c}{\partial z} \quad (21)$$

Transformation of these equations in terms of  $Q$  is readily effected by substitution of

$$L \equiv Qc ; \quad dL/ds = c dQ/ds + Q \partial c / \partial s \quad (22)$$

$$\partial c / \partial s = \cos \gamma \partial c / \partial x + \sin \gamma \partial c / \partial z \quad (23)$$

The transformation yields

$$\left[ \frac{dQ}{ds} \right]_+ = \left[ \frac{dQ}{ds} \right]_- - \frac{Q}{c} \frac{1}{\sin \gamma} \left\{ \left[ \frac{\partial c}{\partial z} \right]_+ - \left[ \frac{\partial c}{\partial z} \right]_- \right\} \quad (24)$$

$$Q_R = -Q_I \quad (25)$$

$$\left[ \frac{dQ}{ds} \right]_R = - \left[ \frac{dQ}{ds} \right]_I + \frac{Q_R}{c} \frac{2}{\sin \gamma_I} \frac{\partial c}{\partial z} \quad (26)$$

### C. Numerical Analogue Design and Accuracy Demands

Consider the design of a numerical analogue for the spreading integration along a single ray, without any regard for consistency with ray spacing. The simplest design is suggested by the Eq. (17) form of the governing differential equation. This design is

$$Q_{\tau+1} + Q_{\tau-1} - 2Q_{\tau} = -\delta t^2 \left[ c^2 \nabla^2 K \right]_{\tau} Q_{\tau} \quad (27)$$

which simplifies to

$$Q_{\tau+1} = A_{\tau} Q_{\tau} - Q_{\tau-1} \quad (28)$$

where

$$A_{\tau} = 2 - \delta t^2 \left[ c^2 v^2 K \right]_{\tau} \quad (29)$$

The constant increment,  $\delta t$ , marks off successive time positions, along the ray, denoted by subscripts,  $0, 1, 2 \dots \tau-1, \tau, \tau+1 \dots$ . The initialization is

$$Q_0 = 0 ; \quad Q_1 = \delta t \quad (30)$$

This analogue scores well in categories of analogy with the differential equation, in stability and simplicity in use. For the purist it has the virtue of reciprocity if the same points--and hence the same values of  $A_{\tau}$ --are used in the reversed propagation. However in applications it leaves us wishing for something extra in the area of accuracy--accuracy, perhaps, beyond the call of duty of a direct numerical analogue.

In sound mapping, at the Fleet Numerical Weather Central, the speed profiles with depth are specified by speed values at whatever depths are felt required for resolution, together with an interpolation formula. The interpolation currently in use is a cubic segment for each depth interval, having gradient and value continuity with the adjoining cubics. While profiles are generally specified miles apart, depths may be specified at intervals in tens of meters, and even in meters. The result is that profiles may have very sharp features in depth. We have seen in

Section IIB that sharp changes in profile gradient cause pronounced changes in the spreading rate. Unless such features are treated as kinks, Eq. (28) must be integrated in very small increments, through these features, in order to contain truncation errors.

This sensitivity would be improved if the numerical analogue could be expressed in terms of perfect differentials, at least in the depth component of the argument. Reduction of the differential order of the speed dependence acts in the same direction of reducing sensitivity.

Either success would also overcome the need for special treatment at internal profile kinks and at reflections (provided we flip the sign of  $L$ ). The consistent scheme that we outline in Section V has such advantages.

### III Single-Profile Simplifications

#### A. Ray Tracing

In the present section we treat the special case of negligible horizontal variation in sound speed. The speed,  $c$ , is prescribed as a function only of depth,  $z$ .

Equation (7), for a ray, may be written

$$\frac{d\gamma}{ds} = - \frac{\cos \gamma}{c} \frac{dc}{dz} \quad (31)$$

Snell's law may be obtained by considering differentials for an increment in depth,  $\delta z$ .

$$\delta\gamma = - \cos \gamma \delta \ln c \frac{\delta s}{\delta z}$$

By substituting  $\sin \gamma = \delta z / \delta s$  we obtain

$$\delta \ln \cos \gamma = \delta \ln c \quad (32)$$

which means that

$$\frac{\cos \gamma}{c} \text{ is constant along a ray.}$$

Referring its value to the reference point  $S$  we may write,

$$\frac{\cos \gamma}{c} = \frac{\cos \gamma_S}{c_S} \quad (33)$$

This is Snell's law, and we have shown that it can include the horizontal coordinate curvature factor by modification of the speed profile.

Snell's law may be useful in ray tracing. It fails to span extremal depths (i.e. over  $\gamma = 0$ ). This limitation arises because multiplication by  $\sin \gamma$  is implicit in its derivation from Eq. (31).

Let us denote successive values along a ray by subscripts . . . .  $m-1, m, m+1$  . . . Equation (32) yields

$$\cos \gamma_{m+1} = \frac{c_{m+1}}{c_m} \cos \gamma_m \quad (34)$$

A form of centered scheme is expressed by

$$\delta x = \frac{1}{2} (\cos \gamma_m + \cos \gamma_{m+1}) \delta s \quad (35)$$

$$\delta z = \frac{1}{2} (\sin \gamma_m + \sin \gamma_{m+1}) \delta s \quad (36)$$

If we choose  $\delta s$  as increment then Eqs. (34), (35) and (36) are an implicit set requiring iteration. If we choose  $\delta z$  as increment their use is explicit.

With  $\delta z$  as increment the integration proceeds nicely until

$$\cos \gamma_{m+1} > 1$$

For such increments,  $m$  to  $m+1$ , we must allow  $\delta z$  to be determined by  $\cos \gamma_{m+1} = 1$ , and the progression in  $z$  is reversed for the subsequent step. At reflections we must also allow  $\delta z$  to be suitably specified.

## B. The Spreading Equation

We specify a point source. Snell's law, expressed by Eq. (33), holds everywhere. The specific wavefront length and the spreading rate at a point on any ray are given by

$$\frac{1}{L} = \frac{\partial \gamma_S}{\partial n} \quad (37)$$

$$\frac{1}{L} \frac{dL}{ds} = \frac{\partial \gamma}{\partial n} \quad (38)$$

We perform the operation  $\partial/\partial n$  on Snell's law at the arbitrary point, and obtain

$$-\frac{\sin \gamma}{c} \frac{\partial \gamma}{\partial n} - \frac{\cos \gamma}{c^2} \frac{\partial c}{\partial n} = -\frac{\sin \gamma_S}{c_S} \frac{\partial \gamma_S}{\partial n} \quad (39)$$

Substitution according to Eqs. (37) and (38) leads to

$$\frac{dL}{ds} = \frac{\sin \gamma_S}{\sin \gamma} \frac{c}{c_S} - \frac{L}{c} \frac{\cos^2 \gamma}{\sin \gamma} \frac{\partial c}{\partial z} \quad (40)$$

This is an order lower than the general form, Eq. (8). Equation (40) fails to integrate through points at which  $\gamma = 0$ , but gives the result at these points that

$$L = \frac{c^2}{c_S} \frac{\sin \gamma_S}{\partial c / \partial z} \quad (41)$$

This suggests that Eq. (40) may have more to offer.

Equation (40) may be cast into a simpler form by substitution of

$$L \equiv \chi \sin \gamma \quad (42)$$

The new principal dependent variable,  $\chi$ , may be interpreted as the horizontal component of the ray spacing. Substitution leads to

$$\frac{d\chi}{ds} = \tan \gamma_S \frac{\cos \gamma}{\sin^2 \gamma} \quad (43)$$

which may also be written

$$\frac{d\chi}{ds} = \frac{c_S \sin \gamma_S c}{c_S^2 - c^2 \cos^2 \gamma_S} \quad (44)$$

This reveals that  $d\chi/ds$ , along a ray, is a function of  $c(z)$  only. We shall use this fact to prove recursion formulae along rays. We shall use the schematic form of Eq. (44),

$$\frac{d\chi}{ds} = F(z) \quad (45)$$

### C. Recursion Formulae

We are still dealing with a single-profile speed distribution,  $c(z)$ . For a ray, defined by  $\gamma_S$ , Eq. (45) may be written

$$\frac{d\chi}{dz} = \frac{F}{\sin \gamma} \quad (46)$$

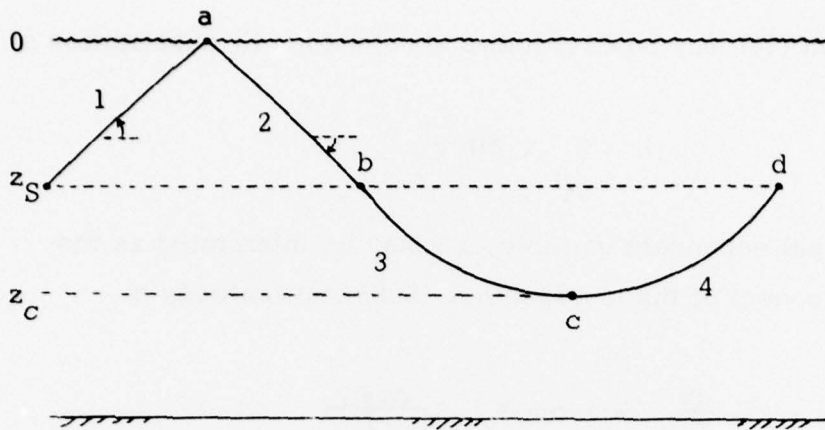


Fig. 2 A full ray cycle

Consider a full cycle of a ray which we segment as indicated in Fig. 2. In segment 1 we have

$$X_1 = \int_{z_S}^z \frac{F}{\sin \gamma_1} dz ; \quad X_a = \int_{z_S}^0 \frac{F}{\sin \gamma_1} dz$$

In segment 2 we have

$$\begin{aligned} X_2 &= X_a + \int_0^z \frac{F}{\sin \gamma_2} dz \\ &= X_a + \int_0^{z_S} \frac{F}{\sin \gamma_2} dz - \int_z^{z_S} \frac{F}{\sin \gamma_2} dz \\ &= X_a + \int_{z_S}^0 \frac{F}{\sin \gamma_1} dz - \int_{z_S}^z \frac{F}{\sin \gamma_1} dz \\ &= 2X_a - X_1 \end{aligned} \tag{47}$$

$$\chi_b = 2\chi_a \quad (48)$$

In segment 3 we have

$$\chi_3 = \chi_b + \int_{z_S}^z \frac{F}{\sin \gamma_3} dz$$

$$\chi_c = \chi_b + \int_{z_S}^{z_C} \frac{F}{\sin \gamma_3} dz$$

In segment 4 we have

$$\chi_4 = \chi_c + \int_{z_C}^z \frac{F}{\sin \gamma_4} dz$$

$$= \chi_c + \int_{z_C}^{z_S} \frac{F}{\sin \gamma_4} dz - \int_z^{z_S} \frac{F}{\sin \gamma_4} dz$$

$$= \chi_c + \int_{z_S}^{z_C} \frac{F}{\sin \gamma_3} dz - \int_{z_S}^z \frac{F}{\sin \gamma_3} dz$$

$$= 2\chi_c - \chi_3 \quad (49)$$

$$\chi_d = 2\chi_c - \chi_b \quad (50)$$

Subsequent cycles have the additive values:  $\chi_d, 2\chi_d, 3\chi_d$ , etc.

These formulae were noted and exploited by E. Hesse of FNWC.

The present verification followed.

#### IV A Scheme for Ray Tracing

For the general case, in which  $c$  varies over the  $x, z$  plane, we would like a scheme that affords consistency with Snell's law as  $\partial c / \partial x \rightarrow 0$ .

We are inclined to prefer  $\delta z$  as the independent increment for the ray integration, except for turning levels ( $\gamma = 0$ ) where  $\delta z$  is determined to the  $\gamma = 0$  level. Also the increment can be adjusted to accommodate the  $z$  levels of interfaces and surface and bottom reflections. Another motivation in specifying  $\delta z$  is that this increment can readily be geared to the variability of the speed structure. For example consider making  $\delta z$  inversely proportional to  $\partial^2 c / \partial z^2$ .

By noting that

$$\frac{\partial}{\partial s} = \cos \gamma \frac{\partial}{\partial x} + \sin \gamma \frac{\partial}{\partial z} \quad (51)$$

$$\frac{\partial}{\partial n} = -\sin \gamma \frac{\partial}{\partial x} + \cos \gamma \frac{\partial}{\partial z} \quad (52)$$

and  $K \equiv \partial c$ , we transform Eq. (7) as follows:

$$\begin{aligned} \frac{d\gamma}{ds} &= \sin \gamma \frac{\partial K}{\partial x} - \cos \gamma \frac{\partial K}{\partial z} \\ - \frac{\sin \gamma}{\cos \gamma} \frac{d\gamma}{ds} &= - \frac{1 - \cos^2 \gamma}{\cos \gamma} \frac{\partial K}{\partial x} + \sin \gamma \frac{\partial K}{\partial z} \\ \frac{d}{ds} \partial n \cos \gamma &= \frac{d}{ds} K - \frac{1}{\cos \gamma} \frac{\partial K}{\partial x} \\ \frac{d}{ds} \partial n \frac{\cos \gamma}{c} &= - \frac{1}{\cos \gamma} \frac{\partial K}{\partial x} \end{aligned} \quad (53)$$

We may develop the numerical analogue in terms of successive points along the ray, . . . . m-1, m, m+1, . . . . Exact application yields

$$\cos \gamma_{m+1} = \frac{c_{m+1}}{c_m} \cos \gamma_m e^{-\int_m^{m+1} \frac{1}{\cos \gamma} \frac{\partial K}{\partial x} ds} \quad (54)$$

which we approximate by

$$\cos \gamma_{m+1} = \frac{c_{m+1}}{c_m} \cos \gamma_m \left[ 1 - \overline{\frac{\delta s}{\cos \gamma} \frac{\partial K}{\partial x}} \right] \quad (55)$$

It remains to decide how to evaluate the term under the curly line. Since it is generally acknowledged that the term is very much smaller than 1 a forward evaluation should be adequate. We substitute

$$\delta s = \delta z / \sin \gamma$$

and obtain

$$\cos \gamma_{m+1} = \frac{c_{m+1}}{c_m} \left[ \cos \gamma_m - \frac{\delta z}{\sin \gamma_m} \left( \frac{\partial K}{\partial x} \right)_m \right] \quad (56)$$

This analogue is, however, not suitable at turning levels ( $\gamma = 0$ ). We design a special operation for the turning increment.

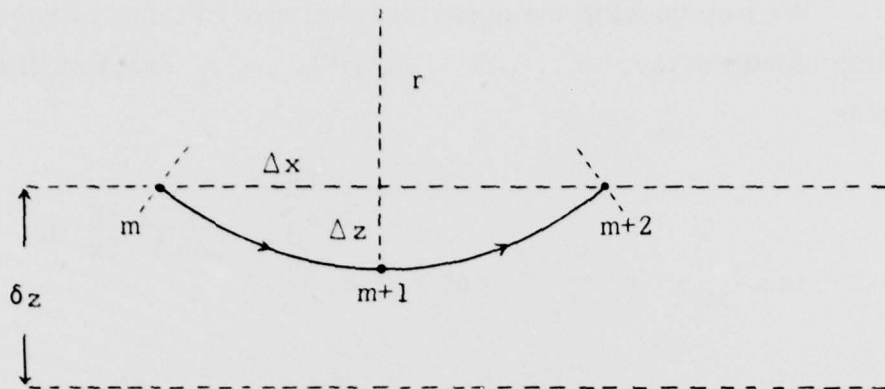


Fig. 3 Integration through  $\gamma = 0$

For increments at the turning level, as shown in Fig. 3, we propose a numerical analogue obtained from

$$\frac{d\gamma}{ds} = -\frac{\partial K}{\partial z} = -\frac{1}{r} \quad (57)$$

That is, we approximate the trace by circle arcs with radius of curvature  $r$ , and obtain

$$\Delta x = \left( \frac{\partial K}{\partial z} \right)^{-1} \sin \gamma_m \quad (58)$$

$$\Delta z = \left( \frac{\partial K}{\partial z} \right)^{-1} (1 - \cos \gamma_m) \quad (59)$$

$$\gamma_{m+1} = 0 \quad (60)$$

$$\gamma_{m+2} = -\gamma_m \quad (61)$$

It may also be strongly argued that the relative contribution of  $\partial K/\partial x$  in Eq. (56) is negligible. If the purist does not agree then he must admit to ranges for which lateral refraction is of comparable significance. At large ranges, it should be realized, concepts of apertures and ducting replace dependence on exact ray positions.

In any case, we have realized consistency with Snell's law by our numerical analogue, Eq. (56). The ray tracing may be performed as follows:

- (1) Specify  $\delta z$  for the increment  $m$  to  $m+1$ , interpolate  $c_{m+1}$  and  $(\partial K/\partial x)_m$ .
- (2) Compute  $\cos \gamma_{m+1}$  from Eq. (56).
- (3) If  $\cos \gamma_{m+1} \geq 1$  abandon the value and use the turning operation defined by Eqs. (58) through (61) to perform two increments to point  $m+2$ .
- (4) If  $\cos \gamma_{m+1} < 1$ , compute  $\sin \gamma_{m+1}$  from

$$\sin \gamma_{m+1} = (1 - \cos^2 \gamma_{m+1})^{1/2} \quad (62)$$

with positive sign for downward progress,  $\delta z > 0$ , and negative for upward progress,  $\delta z < 0$ .

- (5) Compute

$$\delta x = \delta z \frac{\cos \gamma_m + \cos \gamma_{m+1}}{\sin \gamma_m + \sin \gamma_{m+1}} \quad (63)$$

The angle,  $\gamma$ , need not be evaluated.

The remaining details of the ray tracing integration should be straight forward.

## V Adjunct, Consistent Spreading Integration

### A. General Concept

The governing equation, Eq. (8), for the wavefront spreading was derived, in Reference [1], by a combination of the refraction equation,

$$\frac{d\gamma}{ds} = -\frac{1}{c} \frac{\partial c}{\partial n} \quad (64)$$

and the spreading rate,

$$\frac{1}{L} \frac{dL}{ds} = \frac{\partial \gamma}{\partial n} \quad (65)$$

The propagation of a finite segment of wavefront can be depicted by an adequate density of rays each integrated by Eq. (64). The adjunct integration of the spreading must, for consistency, be performed in complete accord.

We propose that the adjunct spreading integration be performed in terms of ambient segments of rays for each increment of the ray tracing. The scheme is depicted in Fig. 4.

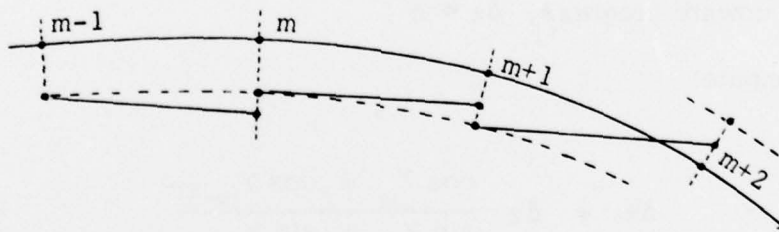


Fig. 4 Adjunct Ray Segments Along a Ray Tracing.  
Normal Segmentation

For each increment of the ray tracing the ambient ray segment is initialized at a point one suitable length unit removed along the normal, on the appropriate side. The side depends on the sign of  $L$ .

The initialization at adjunct point  $m$ , where  $L_m$  and  $(dL/ds)_m$  are known, is

$$\gamma_m^* = \gamma_m + L_m^{-1} (dL/ds)_m \quad (66)$$

The ray segment is calculated by the identical scheme used for the ray. Consider extending the segment to the next ray normal at point  $m+1$ ; it meets this normal at distance  $\delta n_{m+1}$  from the point, with new angle  $\gamma_{m+1}^*$ . From these values compute

$$L_{m+1} = \delta n_{m+1} L_m \quad (67)$$

$$(dL/ds)_{m+1} = L_{m+1} \frac{\gamma_{m+1}^* - \gamma_{m+1}}{\delta n_{m+1}} \quad (68)$$

Note that  $L$  may change sign; the subsequent segment then switches sides.

The scheme may be considered a refinement of using pairs of rays. It is an improvement on pairs because pairs may diverge considerably and become non-representative.

This scheme can be integrated through kinks in the speed profile and at reflections, without special treatment except for flipping the sign of  $L$  at reflections.

## B. Particular Design

Particular design of the adjunct spreading integration must be tied in with the scheme adapted for the ray tracing. We will now go along with the scheme outlined in Section IV.

Depth increment,  $\delta z$ , is used as the independent increment for the ray integration, except for turning levels ( $\gamma = 0$ ). The scheme is depicted in Fig. 5.

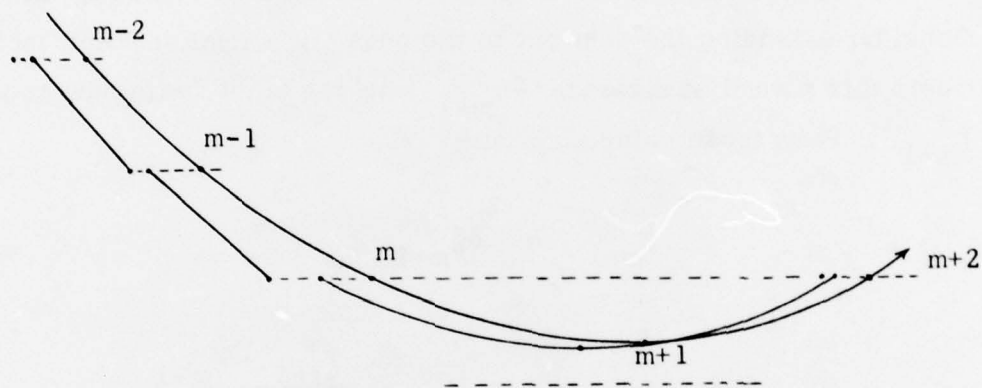


Fig. 5 Adjunct Ray Segments Along a Ray Tracing.  
Horizontal Segmentation

As principal dependent parameter in place of the specific wave-front length we will use

$$\chi \equiv L/\sin \gamma \quad (69)$$

We require one more transformation. Substitution of Eq. (69) into Eq. (65) yields

$$\frac{1}{\chi} \frac{d\chi}{ds} + \frac{\cos \gamma}{\sin \gamma} \frac{\partial \gamma}{\partial s} = \frac{\partial \gamma}{\partial n}$$

$$\frac{\sin \gamma}{\chi} \frac{d\chi}{ds} = -\cos \gamma \frac{\partial \gamma}{\partial s} + \sin \gamma \frac{\partial \gamma}{\partial n}$$

Expansion of operators by Eqs. (51) and (52) leads to

$$\frac{\sin \gamma}{\chi} \frac{d\chi}{ds} = -\frac{\partial \gamma}{\partial x} \quad (70)$$

A more convenient form is

$$\frac{\partial \cos \gamma}{\partial x} = \frac{\sin^2 \gamma}{\chi} \frac{d\chi}{ds} \quad (71)$$

Each adjunct segment is performed as follows. In Fig. 5, at point  $m-2$  for example let  $\chi_{m-2}$  and  $(d\chi/ds)_{m-2}$  already have been computed. The adjunct segment is placed one suitable length unit removed, along minus  $x$  for  $\chi$  positive. The segment is directed according to

$$\cos \gamma_{m-2}^* = \cos \gamma_{m-2} + \frac{\sin^2 \gamma_{m-2}}{\chi_{m-2}} \left( \frac{d\chi}{ds} \right)_{m-2} \quad (72)$$

The segment is extended over increment  $\delta z$ , the same increment used for the ray. Equation (56), applied, yields

$$\cos \gamma_{m-1}^*$$

and Eq. (63) yields  $x_{m-1}^*$  and hence

$$\Delta x_{m-1} = x_{m-1} - x_{m-1}^* \quad (73)$$

From these,

$$X_{m-1} = \Delta x_{m-1} X_{m-2} \quad (74)$$

$$(dX/ds)_{m-1} = \frac{X_{m-1}}{\sin^2 \gamma_{m-1}} \left\{ \cos \gamma_{m-1}^* - \cos \gamma_{m-1} \right\} \quad (75)$$

At the turning levels ( $\gamma = 0$ ) the scheme differs in that the adjunct segment is taken over two increments--i.e. around the extremal--according to Eqs. (57) through (61) as used for the ray itself.

We have probably overlooked some details and complications may remain. However we trust that they can be ironed out in programming and testing.

Acknowledgment — These developments were greatly aided by discussions with Mrs. E. Hesse and others of the Fleet Numerical Weather Central.

## REFERENCES

- [1] "The wavefront divergence factor in ray-intensity integration", M. M. Holl, Meteorology International, Project M-140, Technical Note Two, Contract No. N62271-67-M-2000, April 1967.
- [2] "The reciprocity transformation", M. M. Holl, Meteorology International, Project M-153, Technical Note One, Contract No. N00228-68-C-2406, June 1968.
- [3] "The finite change in wavefront divergence in passing through discontinuities in sound-speed gradient as effected at reflections", M. M. Holl, Meteorology International, Project M-143, Technical Note One, Contract No. N63134-67-M-2854, January 1968.